Tracking on Homogeneous Manifolds

to Andrew Ladd, who had motivated me to study Lie theory

by Jan Ph. Hakenberg[1]

Abstract. We present a computation to yield the transformation that matches best two given sets of landmarks on a linearized homogeneous manifold. The method allows to restrict the set of feasible transformations in a way that is most relevant in practical applications. If the homogeneous manifold is a vector space, the resulting transformation is optimal in the limit.

Keywords: homogeneous manifold, Lie algebra, Campbell-Baker-Hausdorff series, tracking, robotics

1 Motivation

The following excerpt from a recent publication in computer graphics reflects the conception of the tracking problem in engineering.

Quote 1.1. from [MH05], Section 3.3: "Given two sets of points x_i^0 and x_i. Find the rotation matrix R and the translation vectors t and t_0 which minimize

$$\sum_i w_i(R(x_i^0 - t_0) + t - x_i)^2,$$

where the w_i are weights of individual points. In our case, the natural choice for the weights is $w_i = m_i$. The optimal translation vectors turn out to be the center of mass of the initial shape and the center of mass of the actual shape, i.e.

$$t_0 = x_{\mathrm{cm}}^0 = \frac{\sum_i m_i x_i^0}{\sum_i m_i}, \qquad t = x_{\mathrm{cm}} = \frac{\sum_i m_i x_i}{\sum_i m_i},$$

which is physically plausible. Finding the optimal rotation is slightly more involved. Let us define the relative locations $q_i = x_i^0 - x_{\mathrm{cm}}^0$ and $p_i = x_i - x_{\mathrm{cm}}$ of points with respect to their center of mass and let us relax the problem of finding the optimal rotation matrix R to finding the optimal linear transformation A. Now, the term to be minimized is $\sum_i m_i(Aq_i - p_i)^2$. Setting the derivatives with respect to all coefficients of A to zero yields the optimal transformation

$$A = (\sum_i m_i p_i q_i^T)(\sum_i m_i q_i q_i^T)^{-1} = A_{pq} A_{qq}.$$

The second term A_{qq} is a symmetric matrix and, thus, contains only scaling but no rotation. Therefore, the optimal rotation R is the rotational part of A_{pq} which can be found via a polar decomposition $A_{pq} = RS$, where the symmetric part is $S = \sqrt{A_{pq}^T A_{pq}}$ and the rotational part is $R = A_{pq} S^{-1}$." $\diamond$

The landmarks are points x_i^0 and x_i in $\mathbb{R}^3$. We are looking for a transformation (R, t) in the Euclidean group E_3. The idea is to formulate the optimization problem on the homogeneous manifold $\mathbb{R}^3 = \mathrm{E}_3/\mathrm{SO}_3$. Elements of the group E_3 act on $\mathbb{R}^3$ via the left-action.

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2 Propaedeutic

The key ingredient of our tracking method is the Campbell-Baker-Hausdorff series on Lie algebras. The series substitutes the action of a Lie group G with Lie algebra $\mathfrak{g}$ in the vicinity of the neutral element $\mathrm{e} \in G$ via

$$\begin{array}{rcccccc} \circ: & G & \times & G & \rightarrow & G \\ & \downarrow \log & & \downarrow \log & & \uparrow \exp \\ \circ_{\mathrm{cbh}}: & \mathfrak{g} & \times & \mathfrak{g} & \rightarrow & \mathfrak{g} \end{array}$$

If G is a group of matrices, then for a matrix $g \in G$ sufficiently close to the identity matrix $e = I$, and any matrix $X \in \mathfrak{g}$, the functions log and exp are of the form

$$\begin{array}{lcl} \log(I+g) & = & g - \frac{1}{2}g.g + \frac{1}{3}g.g.g - \frac{1}{4}g.g.g.g + \ldots \\ \exp X & = & I + X + \frac{1}{2!}X.X + \frac{1}{3!}X.X.X + \ldots \end{array}$$

The Campbell-Baker-Hausdorff series $\circ_{\mathrm{cbh}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is universal for all Lie algebras, but depends on the commutator tensor $\mathrm{ad}.x.y = [x,y]$ of $\mathfrak{g}$. In [HN91], we find

$$x \circ_{\mathrm{cbh}} y = x + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k}{(k+1)(q_1 + \ldots + q_k + 1)} \frac{(\mathrm{ad}.x)^{p_1}.(\mathrm{ad}.y)^{q_1} \ldots (\mathrm{ad}.x)^{p_k}.(\mathrm{ad}.y)^{q_k}}{p_1! q_1! \cdots p_k! q_k!} \cdot \frac{(\mathrm{ad}.x)^m}{m!}.y$$

The first terms of the series evaluate to

$$x \circ_{\mathrm{cbh}} y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \frac{1}{24}[y,[x,[y,x]]] + \ldots$$

Example 2.1. The Euclidean group $G = \mathrm{E}_2$ is the semi-direct product of planar orientations SO_2 and the vector space $\mathbb{R}^2$. The group is 3-dimensional. Elements $g \in G$ are commonly parametrized using 3×3 homogeneous matrices with the parameters (α, p_x, p_y) for angle and position. Then, the group action is matrix multiplication, and the neutral element is the identity matrix.

The associated Lie algebra $\mathfrak{g} = \mathfrak{e}_2$ is spanned by the matrices $\mathfrak{g} = \langle X_1, X_2, X_3 \rangle$ with the following commutator tensor

$$X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$[,]_{\mathfrak{gl}}$	X_1	X_2	X_3
X_1	0	X_3	$-X_2$
X_2	$-X_3$	0	0
X_3	X_2	0	0

For instance, $[X_1, X_3] = X_1.X_3 - X_3.X_1 = -X_2$.

Any element of the group G is a matrix of the form $\exp \sum_i x_i X_i \in \mathrm{E}_2$. While $x_1 \in \mathbb{R}$ is the angle of rotation, $x_2, x_3 \in \mathbb{R}$ define the amount of translation. The Campbell-Baker-Hausdorff series $\circ_{\mathrm{cbh}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ approximates the combination of two transformations $(x_1, x_2, x_3) \circ (y_1, y_2, y_3)$ by polynomials

$$\begin{array}{rcccccc} \circ: & \exp \sum_i x_i X_i & \times & \exp \sum_i y_i X_i & \rightarrow & \exp \sum_i z_i X_i \\ & \downarrow \log & & \downarrow \log & & \uparrow \exp \\ \circ_{\mathrm{cbh}}: & (x_1, x_2, x_3) & \times & (y_1, y_2, y_3) & \rightarrow & (z_1, z_2, z_3) \end{array}$$

In the evaluation of the series up to order 3, the coefficients relate as

$$
\begin{aligned}
z_1 &= x_1 + y_1\\
z_2 &= x_2 + y_2 + \tfrac{1}{2}(x_3y_1 - x_1y_3) - \tfrac{1}{12}(x_1 - y_1)(x_1y_2 - x_2y_1) + \tfrac{1}{24}x_1y_1(x_3y_1 - x_1y_3)\\
z_3 &= x_3 + y_3 + \tfrac{1}{2}(x_1y_2 - x_2y_1) - \tfrac{1}{12}(x_1 - y_1)(x_1y_3 - x_3y_1) + \tfrac{1}{24}x_1y_1(x_1y_2 - x_2y_1)
\end{aligned}
$$

$\diamond$

Let H be a closed subgroup of G. Then, $M = G/H$ is called a *homogeneous manifold* that comes with two canonic mappings: The *projection* from G to M is given by $\pi : G \to M$ as $g \mapsto gH$. For any element $g \in G$ there exists a unique coset $m = gH \in M$ onto which g projects. The *left-action* of G on M is $\tau : G \times M \to M$ that maps $(g, qH) \mapsto \{g \circ q \circ h : h \in H\} = gqH$.

In practice, the left-action is carried out as

$$
\begin{array}{rccccc}
\tau : & G & \times & M & \to & M\\
& \downarrow \mathrm{Id} & & \downarrow \iota & & \uparrow \pi\\
\circ : & G & \times & G & \to & G
\end{array}
$$

However, the embedding $\iota : M \to G$ is not canonical, and the group action $\circ$ is rarely available in a closed form expression.

To circumvent these difficulties, we implicitely parametrize G by the vector space $\mathfrak{g}$. For a parametrization of $M = G/H$, we decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of $H \subset G$ is uniquely determined. The selection of a vector space complement $\mathfrak{m}$ is simple in practice. The Campbell-Baker-Hausdorff series $\circ_{\mathrm{cbh}}$ recreates the group action $\circ$, as well as the projection π. The previous diagram transcribes into

$$
\begin{array}{rccccc}
\tau : & \mathfrak{g} & \times & \mathfrak{m} & \to & \mathfrak{m}\\
& \downarrow \mathrm{Id} & & \downarrow \iota & & \uparrow \pi\\
\circ_{\mathrm{cbh}} : & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g}
\end{array}
$$

The embeddings $\iota : \mathfrak{u} \to \mathfrak{g}$, and $\iota : \mathfrak{m} \to \mathfrak{g}$ are canonical. For a particular $x \in \mathfrak{g}$ the projection π determines $h \in \mathfrak{h}$ with $x \circ_{\mathrm{cbh}} h \in \mathfrak{m}$. The projection $\pi : \mathfrak{g} \to \mathfrak{m}$ is usually non-linear.

Example 2.2. The Euclidean group $\mathrm{E}_3 = \mathrm{SO}_3 \ltimes \mathbb{R}^3$ encodes orientation and position in 3d space. The group is 6-dimensional. We state the standard representation of the elements $g \in G = \mathrm{E}_3$, as well as the commutator relations of the Lie algebra $\mathfrak{g} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ as

$$
g = \exp \begin{pmatrix} 0 & -x_3 & x_2 & x_4\\ x_3 & 0 & -x_1 & x_5\\ -x_2 & x_1 & 0 & x_6\\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$[,]_{\mathfrak{g}}$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	x_3	$-x_2$	0	x_6	$-x_5$
x_2	$-x_3$	0	x_1	$-x_6$	0	x_4
x_3	x_2	$-x_1$	0	x_5	$-x_4$	0
x_4	0	x_6	$-x_5$	0	0	0
x_5	$-x_6$	0	x_4	0	0	0
x_6	x_5	$-x_4$	0	0	0	0

The vector (x_1, x_2, x_3) is the axis of rotation, whereas (x_4, x_5, x_6) defines the amount of translation. The matrices of E_3 are essential in today's robotics, and computer graphics.

The linear subspace $\mathfrak{h} = \langle x_1, x_2, x_3 \rangle \subset \mathfrak{g}$ is closed with respect to the commutator. $H = \exp \mathfrak{h} = \mathrm{SO}_3$ is a closed subgroup of G. The homogeneous manifold $M = G/H$ is diffeomorphic to $\mathbb{R}^3$.

If we choose $\mathfrak{m} = \langle x_4, x_5, x_6 \rangle$, the Campbell-Baker-Hausdorff series approximates the projection $\pi : \mathfrak{g} \to \mathfrak{m}$ with $\pi(x) = (m_1, m_2, m_3)$ as

$$\begin{aligned}
m_1 &= x_4 + \tfrac{1}{2}(x_2x_6 - x_3x_5) + \tfrac{1}{6}\left(x_1(x_2x_5 + x_3x_6) - x_4(x_2^2 + x_3^2)\right) + \tfrac{1}{24}\left(x_1^2 + x_2^2 + x_3^2\right)(x_3x_5 - x_2x_6) \\
m_2 &= x_5 + \tfrac{1}{2}(x_3x_4 - x_1x_6) + \tfrac{1}{6}\left(x_2(x_3x_6 + x_1x_4) - x_5(x_1^2 + x_3^2)\right) + \tfrac{1}{24}\left(x_1^2 + x_2^2 + x_3^2\right)(x_1x_6 - x_3x_4) \\
m_3 &= x_6 + \tfrac{1}{2}(x_1x_5 - x_2x_4) + \tfrac{1}{6}\left(x_3(x_1x_4 + x_2x_5) - x_6(x_1^2 + x_2^2)\right) + \tfrac{1}{24}\left(x_1^2 + x_2^2 + x_3^2\right)(x_2x_4 - x_1x_5)
\end{aligned}$$

We state the approximation to the left-action $\tau : \mathfrak{g} \times \mathfrak{m} \to \mathfrak{m}$ later in Example 3.4. $\diamond$

3 Tracking on Homogeneous Manifolds

We model the configuration space of a robot and the configuration space of a landmark perceived by the robot in a single Lie group G. The mobility of the robot is characterized by a subset $U \subset G$, which contains the neutral group element $\mathrm{e} \in U$. Any small change in configuration of the robot is determined by an element $u \in U$. If the robot is in configuration $g \in G$, then after a short lapse of time, the configuration is of the form $g \circ u \in G$. Let $\mathfrak{g}$ be the Lie algebra of G. There is a subspace $\mathfrak{u} \subset \mathfrak{g}$ so that $U \subset \exp \mathfrak{u}$. We assume that the configuration space of the landmark is the homogeneous manifold $M = G/H$ where H is a closed subgroup of G.

In the following, we present a strategy to find a transition $u \in U$ of the robot that is consistent with a given transition of a landmark $m \in M$ to $\hat{m} \in M$. We demand $\tau(u, \hat{m}) = m$. As an application, the method could support or even replace the odometry calculation of a robot that is provided with sensors to analyse and correlate the environment.

We implicitely parametrize G by the vector space $\mathfrak{g}$, and M by $\mathfrak{m}$. We employ the Campbell-Baker-Hausdorff series to approximate the mappings $\circ, \tau, \pi$. Now, the equation $\tau(u, \hat{m}) = m$ is with respect to the diagram

$$\begin{array}{rcccccc}
\tau : & \mathfrak{u} & \times & \mathfrak{m} & \to & \mathfrak{m} \\
 & \downarrow \iota & & \downarrow \iota & & \uparrow \pi \\
\circ_{\mathrm{cbh}} : & \mathfrak{g} & \times & \mathfrak{g} & \to & \mathfrak{g}
\end{array}$$

Due to imperfections in the measurements, a perfect match $\tau(u, \hat{m}) = m$ might not exist. A metric $d : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}_0^+$ allows us to pick $u \in \mathfrak{u}$ so that $d(\tau(u, \hat{m}), m)^2$ is minimal.

In practice, there is a number $i = 1, ..., n$ of pairwise correlated landmarks available, where $m^i \in \mathfrak{m}$ moves to $\hat{m}^i \in \mathfrak{m}$. We are interested in $u \in \mathfrak{u}$ that minimizes the sum of the squared errors $e(u) = \sum_i d(\tau(u, \hat{m}^i), m^i)^2$.

Remark 3.1. The algorithm requires some preparations: the commutator tensor ad of the Lie algebra $\mathfrak{g}$, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the vectors that span $\mathfrak{u} \subset \mathfrak{g}$, and the metric d on $\mathfrak{m}$. Then, the input to the method are the landmarks $\{m^i, \hat{m}^i \in \mathfrak{m} : i = 1, ..., n\}$. The algorithm outputs the change in configuration of the robot $u \in \mathfrak{u}$. The corresponding transformation in the Lie group is simply $\exp u \in U$.

In practice, we construct d from a scalar product on $\mathfrak{m}$. We find that Newtons iteration $u^{k+1} = u^k - (\mathrm{d}_u^2 e)^{-1}.\mathrm{d}_u e|_{u^k}$ initialized with $u^0 = 0$ works fine in this case.

In the rare case that $\mathfrak{m} = \mathfrak{g}$, we yield the match simply via $u = \frac{1}{n}\sum_i (m^i \circ_{\mathrm{cbh}} -\hat{m}^i)|_{\mathfrak{u}}$. $\diamond$

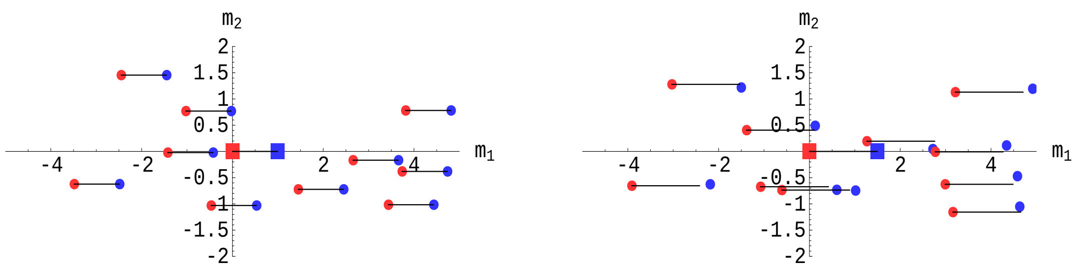

Figure 1 : Set of landmarks perceived by moving train-like robot. Noise is added in the simulation to the right.

Example 3.2. We consider a train-like robot that moves forward and backward on a straight line. To the sides, there are landmarks m^i measured relative to the robot. Based on the perception of the landmarks $\hat{m}^i$ after a short lapse of time, we wish to estimate the distance u travelled by the robot.

The configuration space of the robot is $\mathbb{R}$, while the configuration space of each landmark is $\mathbb{R}^2$. Thus, we model the scenario in the Lie-group $G = \mathbb{R}^2$. The group action is simply vector addition. The neutral element is $\mathrm{e} = (0, 0)$.

We assume the robot moves along the x_1-axis. Then, the mobility is $U = \{(x_1, 0) : x_1 \in \mathbb{R}\}$. Since the configuration space of a landmark is already the full group $G = \mathbb{R}^2$, we fix $H = \{\mathrm{e}\}$, and $M = G/\{\mathrm{e}\} = G$.

The Lie algebra of G is the vector space $\mathfrak{g} = \mathbb{R}^2$. The Campbell-Baker-Hausdorff series $\circ_{\mathrm{cbh}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ simply maps $(x_1, x_2) \circ_{\mathrm{cbh}} (y_1, y_2) \mapsto (x_1 + y_1, x_2 + y_2)$. The embeding $\iota : \mathfrak{u} \to \mathfrak{g}$ is $(u_1) \mapsto (u_1, 0)$. Because $\mathfrak{m} = \mathfrak{g}$, the projection $\pi : \mathfrak{g} \to \mathfrak{m}$ is the identity.

With the Euclidean distance as a metric on $\mathfrak{m}$, the problem reduces to find $u_1 \in \mathbb{R}$ that minimizes

$$e(u_1) = \sum d\big((u_1, 0) + \hat{m}^i, m^i\big)^2 = \sum (u_1 + \hat{m}^i_1 - m^i_1)^2 + (\hat{m}^i_2 - m^i_2)^2$$

The solution is $u_1 = \frac{1}{n} \sum_i (m^i_1 - \hat{m}^i_1)$. Much Ado about Nothing. $\diamond$

Example 3.3. The configuration space of an ordinary differential-drive robot is the Lie group $G = \mathrm{E}_2$. An element $g \in G$ encodes the orientation of the axis and the global position of the robot in the plane $\mathbb{R}^2$. The differential-drive allows the robot to simultaneously move forward and turn. However, the robot does not slide sidewards. Thus, the mobility is generated by $\mathfrak{u} = \langle X_1, X_2 \rangle$ as $U \subset \exp u_1 X_1 + u_2 X_2$ for coefficients $u_1, u_2 \in \mathbb{R}$. The matrices X_j are from Example 2.1.

We assume that a landmark $m \in \mathbb{R}^2$ resides in the plane, and that it is perceived at $\hat{m}$, after the robot has moved by $u \in U$. The configuration space of the landmark is $M = G/H$, where $H = \exp X_1 \subset G$ is a closed subgroup of G. Then $\mathfrak{h} = \langle X_1 \rangle$, and we define $\mathfrak{m} = \langle X_2, X_3 \rangle$.

We aim to compute τ in the following diagram.

$$\begin{array}{rccccc}
\tau : & \mathfrak{u} & \times & \mathfrak{m} & \longrightarrow & \mathfrak{m} \\
 & \downarrow \iota & & \downarrow \iota & & \uparrow \pi \\
\circ_{\mathrm{cbh}} : & \mathfrak{g} & \times & \mathfrak{g} & \longrightarrow & \mathfrak{g}
\end{array}
\qquad
\begin{array}{rccccc}
\tau : & (u_1, u_2) & \times & (\hat{m}_1, \hat{m}_2) & \longrightarrow & (m_1, m_2) \\
 & \downarrow \iota & & \downarrow \iota & & \uparrow \pi \\
\circ_{\mathrm{cbh}} : & (u_1, u_2, 0) & \times & (0, \hat{m}_1, \hat{m}_2) & \longrightarrow & (x_1, x_2, x_3)
\end{array}$$

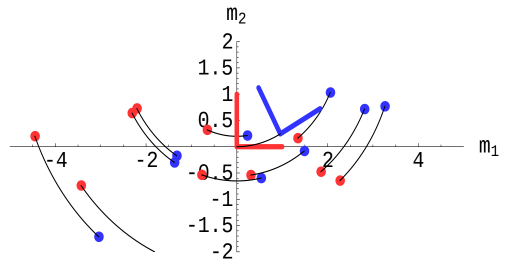

Figure 2 : Set of landmarks tracked by moving differential-drive robot. Noise is added in the simulation to the right.

The embeddings ι are canonic. The Campbell-Baker-Hausdorff series specific to E_2 is stated in Example 2.1. We obtain the projection $\pi : \mathfrak{g} \to \mathfrak{m}$ by $(x_1, x_2, x_3) \circ_{\mathrm{cbh}} (-x_1, 0, 0) = (0, m_1, m_2)$, which is approximately

$$\begin{aligned} m_1 &= x_2 - \tfrac{1}{2}x_1x_3 - \tfrac{1}{6}x_1^2x_2 + \tfrac{1}{24}x_1^3x_3 + \tfrac{1}{120}x_1^4x_2 - \tfrac{1}{720}x_1^5x_3 - \tfrac{1}{5040}x_1^6x_2 + \tfrac{1}{40320}x_1^7x_3 + \tfrac{1}{362880}x_1^8x_2 \\ m_2 &= x_3 + \tfrac{1}{2}x_1x_2 - \tfrac{1}{6}x_1^2x_3 - \tfrac{1}{24}x_1^3x_2 + \tfrac{1}{120}x_1^4x_3 + \tfrac{1}{720}x_1^5x_2 - \tfrac{1}{5040}x_1^6x_3 - \tfrac{1}{40320}x_1^7x_2 + \tfrac{1}{362880}x_1^8x_3 \end{aligned}$$

The mapping τ formalizes the influence of the turning u_1 and forward motion u_2 on the perception of the landmark at position $(\hat{m}_1, \hat{m}_2)$. The result is approximately

$$\begin{aligned} m_1 &= \hat{m}_1 + u_2 - u_1\hat{m}_2 - \tfrac{1}{2}u_1^2\hat{m}_1 + \tfrac{1}{6}(u_1^3\hat{m}_2 - u_1^2u_2) + \tfrac{1}{24}u_1^4\hat{m}_1 + \tfrac{1}{120}u_1^4u_2 - \tfrac{1}{144}u_1^5\hat{m}_2 - \tfrac{1}{2160}u_1^6\hat{m}_1 \\ m_2 &= \hat{m}_2 + u_1\hat{m}_1 + \tfrac{1}{2}(u_1u_2 - u_1^2\hat{m}_2) - \tfrac{1}{6}u_1^3\hat{m}_1 + \tfrac{1}{24}(u_1^4\hat{m}_2 - u_1^3u_2) + \tfrac{1}{144}u_1^5\hat{m}_1 - \tfrac{1}{2160}u_1^6\hat{m}_2 \end{aligned}$$

For instance, if the robot perceives the transition of a landmark

$$\text{from} \quad m = (1.4867, -0.0808) \in \mathbb{R}^2 \quad \text{to} \quad \hat{m} = (0.3071, -0.5388) \in \mathbb{R}^2,$$

the equations imply (using Newtons iteration) that the robot has moved by $u = (0.5, 1) \in \mathfrak{u}$. The associated transformation matrix is simply

$$\exp u_1X_1 + u_2X_2 = \exp \begin{pmatrix} 0 & -0.5 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.8775 & -0.4794 & 0.9588 \\ 0.4794 & 0.8775 & 0.2448 \\ 0 & 0 & 1 \end{pmatrix}$$

Given a set of pairwise correlated landmarks $m^i, \hat{m}^i \in \mathbb{R}^2$ for $i = 1, ..., n$ and a metric d on $\mathfrak{m}$, we ask for $u \in \mathfrak{u}$ that minimizes $e(u) = \sum_i d\big(\tau(u, \hat{m}^i), m^i\big)^2$. Since $\mathbb{R}^2$ is a vector space, the Euclidean metric is a natural choice. In Remark 3.5, we show how the order of the series expansion affects the accuracy. $\diamond$

While a common laser range finder detects landmarks in the plane $\mathbb{R}^2$, stereo cameras correlate landmarks in their field of view to locate them in $\mathbb{R}^3$. The next example illustrates how to derive the motion of a robot in 3d space from the transition of landmarks in $\mathbb{R}^3$.

We demonstrate that the Campbell-Baker-Hausdorff series substitutes quaternions, Euler-angles, and polar decomposition. The formula eliminates the drawback of excessive parameters, and obeys the symmetry of the problem. Moreover, we are able to account the mobility of the robot.

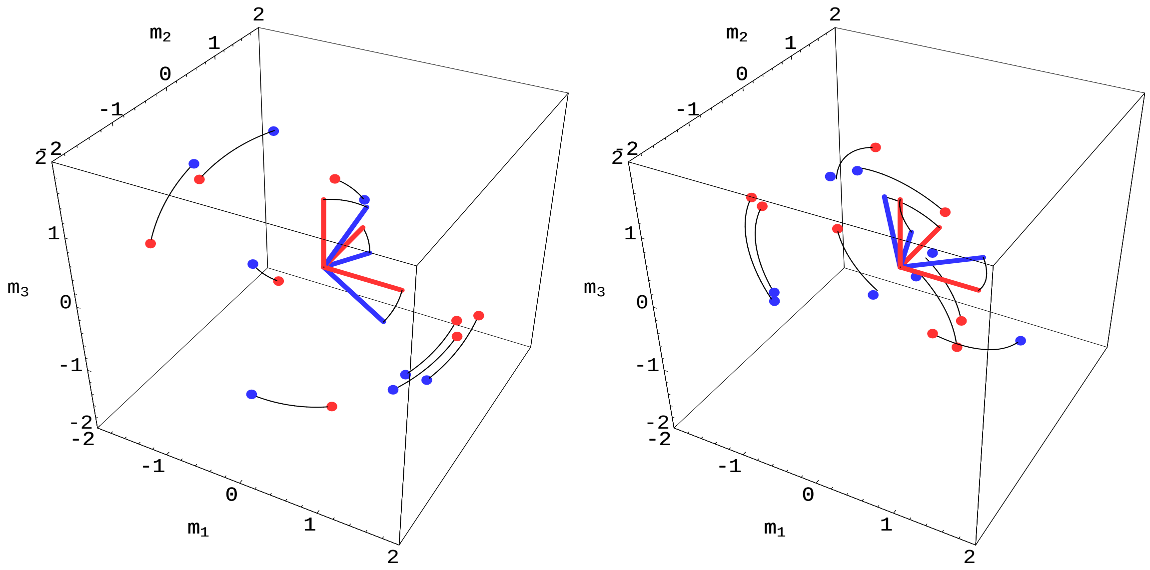

Figure 3 : Set of landmarks perceived by rotating robot with $u_4 = u_5 = u_6 = 0$. Noise is added in the simulation to the right.

Example 3.4. Let the configuration space of our robot be the 6-dimensional Euclidean group $G = \mathrm{E}_3$ that was introduced in Example 2.2. An element of the group encodes orientation and position in 3d space.

Let the configuration space of the landmark be $\mathbb{R}^3$. With $H = \mathrm{SO}_3 \subset G$ as the subgroup of orientations, $M = G/H$ is diffeomorphic to the configuration space of the landmarks $\mathbb{R}^3$.

The Lie algebra decomposes into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. At first, we shall assume $\mathfrak{u} = \mathfrak{g}$, that the robot moves with 6 degrees of freedom. Then, we approximate $\tau : \mathfrak{u} \times \mathfrak{m} \to \mathfrak{m}$ with $\tau(u, \hat{m}) = m$ as

$$\begin{aligned}
m_1 &= u_4 + \hat{m}_1 - \hat{m}_2 u_3 + \hat{m}_3 u_2 + \tfrac{1}{2}(u_2 u_6 - u_3 u_5) + \tfrac{1}{4}(-\hat{m}_1(u_2^2 + u_3^2) + \hat{m}_2 u_1 u_2 + \hat{m}_3 u_1 u_3) \\
m_2 &= u_5 + \hat{m}_1 u_3 + \hat{m}_2 - \hat{m}_3 u_1 + \tfrac{1}{2}(u_3 u_4 - u_1 u_6) + \tfrac{1}{4}(+\hat{m}_1 u_1 u_2 - \hat{m}_2(u_1^2 + u_3^2) + \hat{m}_3 u_2 u_3) \\
m_3 &= u_6 - \hat{m}_1 u_2 + \hat{m}_2 u_1 + \hat{m}_3 + \tfrac{1}{2}(u_1 u_5 - u_2 u_4) + \tfrac{1}{4}(+\hat{m}_1 u_1 u_3 + \hat{m}_2 u_2 u_3 - \hat{m}_3(u_1^2 + u_2^2))
\end{aligned}$$

Depending on the characteristics of the robot and its environment, we are free to restrict the mobility by setting $u_j = 0$ for several $j \in \{1, ..., 6\}$. In the illustrations, we choose three different combinations. $\diamond$

Remark 3.5. In the Examples 3.3 and 3.4, expansions of the Campbell-Baker-Hausdorff series of order 4 give accurate results.

Ex.	order	u_1	u_2
3.3	2	0.494481	0.99826
	3	0.499927	1.00088
	4	0.500048	1.00002
	∞	0.5	1.0

Ex.	order	u_1	u_2	u_3
3.4	2	-0.295240	0.492529	0.197448
	3	-0.299861	0.499874	0.199921
	4	-0.300054	0.500104	0.200037
	∞	-0.3	0.5	0.2

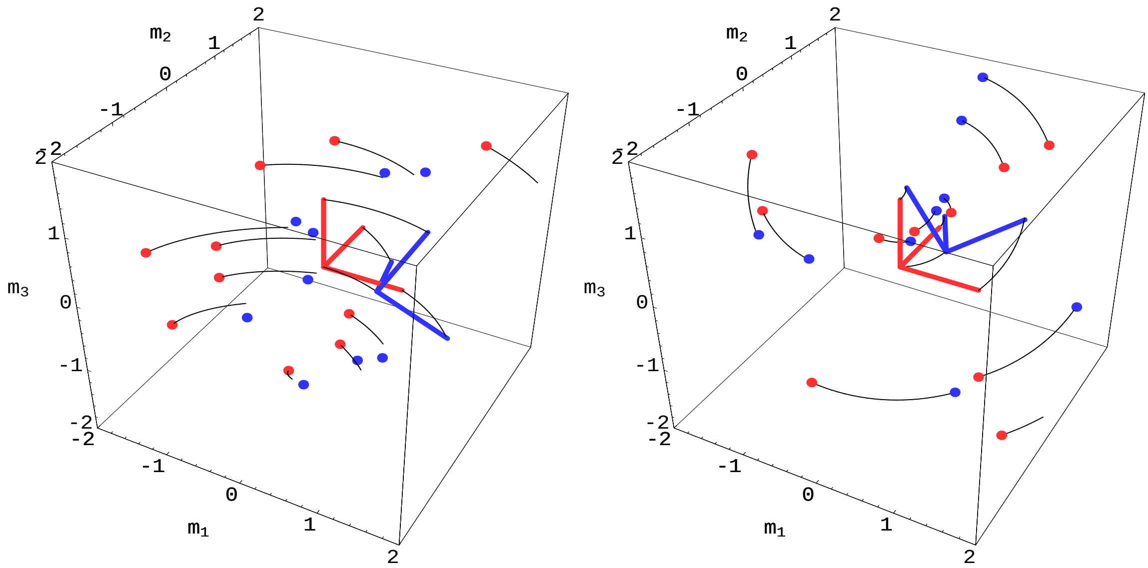

Figure 4 : Set of landmarks perceived by rotating and translating robot. To the left, we have set $u_5 = u_6 = 0$. In the right graphic, $u_1 = u_6 = 0$ holds.

4 Outlook

The convergence radius of the Campbell-Baker-Hausdorff series is finite in most instances. Then, the method can identify transformations $u \in U$ only in the vicinity of the identity. However, in tracking applications with a sufficiently high sampling rate, our method is feasible.

The configuration space of a landmark is a vector space $M = \mathbb{R}^n$ in all examples that we have encountered. Otherwise, the Euclidean metric on $\mathfrak{m}$ is not the canonic choice to measure the distance between two points $m, \hat{m} \in \mathfrak{m}$. With more effort, one adapts the method to match landmarks on "non-linear" homogeneous manifolds such as the 2-dimensional sphere $S^2 = \mathrm{SO}_3/\mathrm{SO}_2$ with an appropriate metric.

5 Bibliography

[DK95] B. Doubrov, B. Komrakov: *Low-dimensional Pseudo-Riemannian Homogeneous Spaces*, Preprint University of Oslo, No. 13, March 1995

[Ha08] J. Hakenberg: *Tracking on Homogeneous Manifolds: Examples and Illustrations (Mathematica 5.2)*, `www.hakenberg.org`

[HN91] J. Hilgert, K. Neeb: *Lie-Gruppen und Lie-Algebren*, Vieweg, 1991

[MH05] M. Müller, B. Heidelberger, M. Teschner, M. Gross: *Meshless Deformations Based on Shape Matching*, Proceedings of SIGGRAPH'05, pp. 471-478

[Ne04] K. Neeb: *Skriptum zur Vorlesung Lie-Gruppen*, available online, 2004